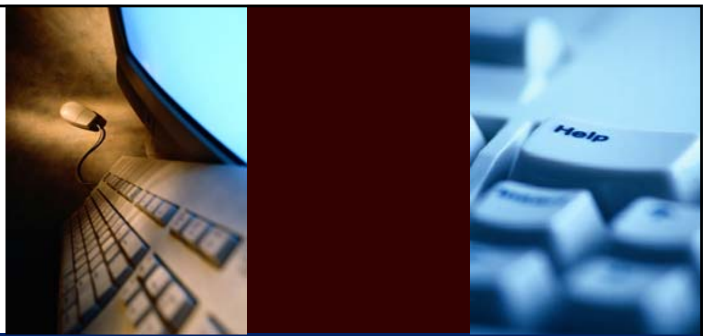


Optimización

Ronald Cuela

Contenido

Maximización Estática

❖ **Problema general sin restricciones**

$$\underset{x_i}{Max} \ f(x_1, x_2, ..., x_n) \quad \rightarrow \quad i = 1,2,...,n$$

Condiciones de Primer Orden:

$$\frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_i} = 0 \quad \rightarrow \quad i = 1,2,...,n$$

Condiciones de Segundo Orden:

$$H\left[f(x_1, x_2, ..., x_n) \right]_{x=x^*}$$

Maximización Estática

❖ **Problema general con restricciones de igualdad**

$$\underset{x_i}{Max} \ f(x_1, x_2, ..., x_n) \quad \rightarrow \quad i = 1,2,...,n$$

$$s.a.: \quad g_j(x_1, x_2, ..., x_n) = 0 \quad \rightarrow \quad j = 1,2,...,m$$

Condiciones de Primer Orden: $L = f(x_i) + \sum_{j=1}^{m} \lambda_j g_j(x_i)$

$$\frac{\partial L}{\partial x_i} = \frac{\partial f(x_i)}{\partial x_i} + \sum_{j=1}^{m} \left[\lambda_j \frac{\partial g_j(x_i)}{\partial x_i} \right] = 0 \quad \rightarrow \quad i = 1,2,...,n$$

$$\frac{\partial L}{\partial \lambda_j} = g_j(x_1, x_2, ..., x_n) = 0 \quad \rightarrow \quad j = 1,2,...,m$$

Condiciones de Segundo Orden:

$$H\left[f(x_1, x_2, ..., x_n) \right]_{x=x^*}$$

$$H\left[g(x_1, x_2, ..., x_m) \right]_{x=x^*}$$

Maximización Estática

❖ **Problema general con restricciones de desigualdad**

$$\underset{x_i}{Max} \ f(x_1, x_2, ..., x_n) \ \rightarrow \ i = 1,2,...,n$$

$$s.a.: \ g_j(x_1, x_2, ..., x_n) \geq 0 \ \rightarrow \ j = 1,2,...,m$$

Condiciones de Primer Orden: $L = f(x_i) + \sum_{j=1}^{m} \lambda_j g_j(x_i)$

$$\frac{\partial L}{\partial x_i} = 0 \ \rightarrow \ i = 1,2,...,n$$

$$\lambda_j \geq 0 \qquad \frac{\partial L}{\partial \lambda_j} \geq 0 \qquad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0 \ \rightarrow \ j = 1,2,...,m$$

Condiciones de Segundo Orden:

$$H[f(x_1, x_2, ..., x_n)]_{x=x^*}$$

$$H[g(x_1, x_2, ..., x_m)]_{x=x^*}$$

Maximización Estática

❖ **Problema general con restricciones de desigualdad y no negatividad**

$$\underset{x_i}{Max} \ f(x_1, x_2, ..., x_n) \ \rightarrow \ i = 1,2,...,n$$

$$s.a.: \ g_j(x_1, x_2, ..., x_n) \geq 0 \ \rightarrow \ j = 1,2,...,m$$

$$x_i \geq 0 \ \rightarrow \ i = 1,2,...,n$$

Condiciones de Primer Orden: $L = f(x_i) + \sum_{j=1}^{m} \lambda_j g_j(x_i)$

$$x_i \geq 0 \qquad \frac{\partial L}{\partial x_i} \leq 0 \qquad x_i \frac{\partial L}{\partial x_i} = 0 \ \rightarrow \ i = 1,2,...,n$$

$$\lambda_j \geq 0 \qquad \frac{\partial L}{\partial \lambda_j} \geq 0 \qquad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0 \ \rightarrow \ j = 1,2,...,m$$

Condiciones de Segundo Orden:

$$H[f(x_1, x_2, ..., x_n)]_{x=x^*}$$

$$H[g(x_1, x_2, ..., x_m)]_{x=x^*}$$

Teoría Macrodinámica – Ronald Cuela

Contenido

Cálculo de variaciones

❖ **Problema general**

$$MaxV = \int_0^T f(t, y_t, \dot{y}_t)\partial t$$

$$s.a.: \quad y(0) = y_0$$

$$y(T) = y_T$$

Condiciones de Primer Orden:

$$f_y = \frac{\partial f_{\dot{y}}}{\partial t}$$

Condiciones de Segundo Orden:

$$H\left[f(t, y_t, \dot{y}_t)\right]_{(y^*, \dot{y}^*)}$$

Cálculo de variaciones

❖ **Condición de transversalidad**

$$\left\lfloor f_{\dot{y}} \right\rfloor_{t=T} \Delta y_T + \left\lfloor f - \dot{y}\ f_{\dot{y}} \right\rfloor_{t=T} \Delta T = 0$$

Horizonte temporal fijo:

$$\left\lfloor f_{\dot{y}} \right\rfloor_{t=T} = 0$$

Valor terminal fijo:

$$\left\lfloor f - \dot{y}\ f_{\dot{y}} \right\rfloor_{t=T} = 0$$

Curva terminal:

$$\left\lfloor f_{\dot{y}} \right\rfloor_{t=T} g'(T) + \left\lfloor f - \dot{y}\ f_{\dot{y}} \right\rfloor_{t=T} = 0$$

Contenido

1. Maximización Estática
2. Cálculo de Variaciones
3. Control Óptimo
4. Programación Dinámica

Control Óptimo

- **Problema general**

$$MaxV = \int_0^T f(t, y_t, u_t)\partial t$$

$$s.a.: \quad \dot{y}_t = g(t, y_t, u_t)$$

$$y(0) = y_0$$

$$y(T) = y_T$$

Condiciones de Primer Orden: $H(t, y_t, u_t) = f(t, y_t, u_t) + \lambda_t g(t, y_t, u_t)$

$$\underset{u_t}{Max}\, H(t, y_t, u_t)$$

$$\frac{\partial H}{\partial y_t} = -\dot{\lambda}_t$$

$$\frac{\partial H}{\partial \lambda_t} = \dot{y}_t$$

Condiciones de Segundo Orden: $Hes[H(t, y_t, u_t)]_{(y^*, u^*)}$

Control Óptimo

- **Condición de transversalidad**

$$[H]_{t=T}\Delta T - \lambda_T \Delta y_T = 0$$

Horizonte temporal fijo:

$$\lambda_T = 0$$

Valor terminal fijo:

$$[H]_{t=T} = 0$$

Curva terminal:

$$[H]_{t=T} - \lambda_T g'(T) = 0$$

Control Óptimo

❖ **Ejemplo**

$$MaxV = \int_0^T Ln\ c_t \partial t$$

$$s.a.: \quad \dot{s}_t = rs_t - c_t$$

$$s(0) = s_0$$

$$s(T) = s_T$$

Condiciones de Primer Orden: $H(t, s_t, c_t) = Ln\ c_t + \lambda_t (rs_t - c_t)$

$\underset{c_t}{Max}\, H \quad \rightarrow \quad \frac{1}{c_t} - \lambda_t = 0$

$\frac{\partial H}{\partial s_t} = -\dot{\lambda}_t \quad \rightarrow \quad r\lambda_t = -\dot{\lambda}_t$

$\frac{\partial H}{\partial \lambda_t} = \dot{s}_t \quad \rightarrow \quad rs_t - c_t = \dot{s}_t$

Control Óptimo

❖ **Ejemplo**

Resultado:

$$s_t = s_0 e^{rt} - \left(\frac{s_0 - s_T e^{-rT}}{T} \right) t e^{rt}$$

$$c_t = \left(\frac{s_0 - s_T e^{-rT}}{T} \right) e^{rt}$$

$$\lambda_t = \left(\frac{T}{s_0 - s_T e^{-rT}} \right) e^{-rt}$$

Condiciones de Segundo Orden:

$$Hes[H(t, y_t, u_t)]_{(y^*, u^*)} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{c_t^2} \end{bmatrix}$$

Contenido

Programación Dinámica

- **Problema general**

$$Max\ V_t = \sum_{t=0}^{T} f_t(y_t, u_t) + Z(y_{T+1})$$

$$s.a.: \quad y_{t+1} \leq g_t(y_t, u_t) \qquad \forall t = 0,1,\ldots,T$$

$$y(0) = y_0$$

$$y(T+1) = y_{T+1}$$

$$u_t, y_t \geq 0$$

Condiciones de Primer Orden: $L = \sum_{t=0}^{T} f_t(y_t, u_t) + Z(y_{T+1}) + \sum_{t=0}^{T} \lambda_t (g_t(y_t, u_t) - y_{t+1})$

$$\frac{\partial L}{\partial u_t} \leq 0 \qquad u_t \geq 0 \qquad u_t \frac{\partial L}{\partial u_t} = 0$$

$$\frac{\partial L}{\partial y_{t+1}} \leq 0 \qquad y_{t+1} \geq 0 \qquad y_{t+1} \frac{\partial L}{\partial y_{t+1}} = 0$$

$$\frac{\partial L}{\partial \lambda_t} \leq 0 \qquad \lambda_t \geq 0 \qquad \lambda_t \frac{\partial L}{\partial \lambda_t} = 0$$

Programación Dinámica

❖ Ecuación de Euler

Asumiendo solución interna

$$L=\sum_{t=0}^{T} f_t(y_t,u_t)+Z(y_{T+1})+\sum_{t=0}^{T}\lambda_t(g_t(y_t,u_t)-y_{t+1})$$

$$\frac{\partial L}{\partial u_t}=0 \rightarrow \frac{\partial L}{\partial u_t}=\frac{\partial f_t}{\partial u_t}+\lambda_t\frac{\partial g_t}{\partial u_t}=0 \rightarrow \lambda_t=-\frac{\frac{\partial f_t}{\partial u_t}}{\frac{\partial g_t}{\partial u_t}} \qquad \lambda_{t+1}=-\frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}}$$

$$\frac{\partial L}{\partial y_{t+1}}=0 \rightarrow \frac{\partial L}{\partial y_{t+1}}=\frac{\partial f_{t+1}}{\partial y_{t+1}}+\lambda_{t+1}\frac{\partial g_{t+1}}{\partial y_{t+1}}-\lambda_t=0 \rightarrow \lambda_t=\frac{\partial f_{t+1}}{\partial y_{t+1}}+\lambda_{t+1}\frac{\partial g_{t+1}}{\partial y_{t+1}}$$

$$\frac{\partial L}{\partial \lambda_t}=0 \rightarrow \frac{\partial L}{\partial \lambda_t}=g_t(y_t,u_t)-y_{t+1}=0$$

Reemplazando en la primera condición

$$\frac{\partial f_t}{\partial u_t}+\frac{\partial g_t}{\partial u_t}\left[\frac{\partial f_{t+1}}{\partial y_{t+1}}-\frac{\partial g_{t+1}}{\partial y_{t+1}}\frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}}\right]=0$$

Programación Dinámica

❖ Problema general

$$Max\ V_t=\sum_{t=0}^{T} f_t(y_t,u_t)+Z(y_{T+1})$$

$$s.a.:\quad y_{t+1}\le g_t(y_t,u_t) \qquad \forall t=0,1,\ldots,T$$

$$y(0)=y_0$$

$$y(T+1)=y_{T+1}$$

$$u_t,y_t\ge 0$$

Condiciones de Segundo Orden:

$$Hes[f_t(y_t,u_t)]_{(y^*,u^*)}$$

$$Hes[g_t(y_t,u_t)]_{(y^*,u^*)}$$

Programación Dinámica

❖ Principio de optimalidad

$$u_0^*,\ u_1^*,\ldots,\ u_t^*,\ u_{t+1}^*,\ldots,\ u_{T-1}^*,\ u_T^*$$

$$Max\ V_t = \sum_{t=0}^{T} f_t(y_t,u_t) + Z(y_{T+1})$$

$$s.a.:\quad y_{t+1} \le g_t(y_t,u_t) \qquad \forall t = 0,1,\ldots,T$$

$$y_0\ dado$$

$$y_{T+1}\ dado$$

$$Max\ V_t = \sum_{\tau=t}^{T} f_\tau(y_\tau,u_\tau) + Z(y_{T+1})$$

$$s.a.:\quad y_{\tau+1} \le g_\tau(y_\tau,u_\tau) \qquad \forall \tau = t,t+1,\ldots,T$$

$$y_t\ dado$$

$$y_{T+1}\ dado$$

Programación Dinámica

❖ Ecuación de Bellman

$$Max\ V_t = \sum_{\tau=t}^{T} f_\tau(y_\tau,u_\tau) + Z(y_{T+1})$$

$$s.a.:\quad y_{\tau+1} \le g_\tau(y_\tau,u_\tau) \qquad \forall \tau = t,t+1,\ldots,T$$

$$y_t\ dado$$

$$y_{T+1}\ dado$$

$$V_t(y_t) = \underset{u_t}{Max}\left[f_t(y_t,u_t) + V_{t+1}(y_{t+1})\right]$$

$$s.a.:\quad y_{t+1} \le g_t(y_t,u_t) \qquad \forall t = 0,1,\ldots,T$$

$$y_0\ dado$$

$$y_{T+1}\ dado$$

❖ Ecuación de Benveniste y Scheinkman

De Bellman

$$\frac{\partial f_t}{\partial u_t} + \frac{\partial V_{t+1}}{\partial y_{t+1}}\frac{\partial g_t}{\partial u_t} = 0$$

$$y_{t+1} = g_t(y_t,u_t)$$

Función de política

$$u_t = h(y_t)$$

Teorema de la envolvente

$$V_t(y_t) = f_t(y_t,h(y_t)) + V_{t+1}(g_t(y_t,h(y_t)))$$

$$\frac{\partial V_t}{\partial y_t} = \frac{\partial f_t}{\partial y_t} + \frac{\partial V_{t+1}}{\partial y_{t+1}}\frac{\partial g_t}{\partial y_t}$$

❖ **Ecuación de Euler**

De la CPO de Bellman

$$\frac{\partial f_t}{\partial u_t}+\frac{\partial V_{t+1}}{\partial y_{t+1}}\frac{\partial g_t}{\partial u_t}=0 \rightarrow \frac{\partial V_{t+1}}{\partial y_{t+1}}=-\frac{\frac{\partial f_t}{\partial u_t}}{\frac{\partial g_t}{\partial u_t}} \rightarrow \frac{\partial V_{t+2}}{\partial y_{t+2}}=-\frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}}$$

De Benveniste y Scheinkman

$$\frac{\partial V_t}{\partial y_t}=\frac{\partial f_t}{\partial y_t}+\frac{\partial V_{t+1}}{\partial y_{t+1}}\frac{\partial g_t}{\partial y_t} \rightarrow \frac{\partial V_{t+1}}{\partial y_{t+1}}=\frac{\partial f_{t+1}}{\partial y_{t+1}}+\frac{\partial V_{t+2}}{\partial y_{t+2}}\frac{\partial g_{t+1}}{\partial y_{t+1}}$$

En la CPO

$$\frac{\partial f_t}{\partial u_t}+\frac{\partial g_t}{\partial u_t}\left[\frac{\partial f_{t+1}}{\partial y_{t+1}}-\frac{\partial g_{t+1}}{\partial y_{t+1}}\frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}}\right]=0$$

❖ **Ejemplo:**

$$Max\ V_t=\sum_{t=0}^{T}\beta^t Ln\ c_t$$

$$s.a.:\quad s_{t+1}\le(1+r)(s_t-c_t)\qquad \forall t=0,1,\ldots,T$$

$$s(0)=s_0$$

$$s_{T+1}\ \ libre$$

$$c_t, s_t \ge 0$$

Ecuación de Euler:

$$\frac{\partial f_t}{\partial u_t}+\frac{\partial g_t}{\partial u_t}\left[\frac{\partial f_{t+1}}{\partial y_{t+1}}-\frac{\partial g_{t+1}}{\partial y_{t+1}}\frac{\frac{\partial f_{t+1}}{\partial u_{t+1}}}{\frac{\partial g_{t+1}}{\partial u_{t+1}}}\right]=0$$

$$\frac{\beta^t}{c_t}-(1+r)\left[0+(1+r)\frac{\frac{\beta^{t+1}}{c_{t+1}}}{(1+r)}\right]=0$$

$$c_{t+1}=\beta(1+r)c_t$$

Programación Dinámica

❖ **Ejemplo:** $s_{t+1} = (1+r)(s_t - c_t)$ ...Restricción

$c_{t+1} = \beta(1+r)c_t$ ...Ecuación de Euler

Función de Política: $c_t = h(s_t)$

En T: $s_{T+1} = (1+r)(s_T - c_T) = 0$ $\qquad$ $c_T = s_T$

En T-1: $s_T = (1+r)(s_{T-1} - c_{T-1})$

$c_T = \beta(1+r)c_{T-1}$

$c_T = (1+r)(s_{T-1} - c_{T-1})$

$$c_{T-1} = \frac{s_{T-1}}{1+\beta}$$

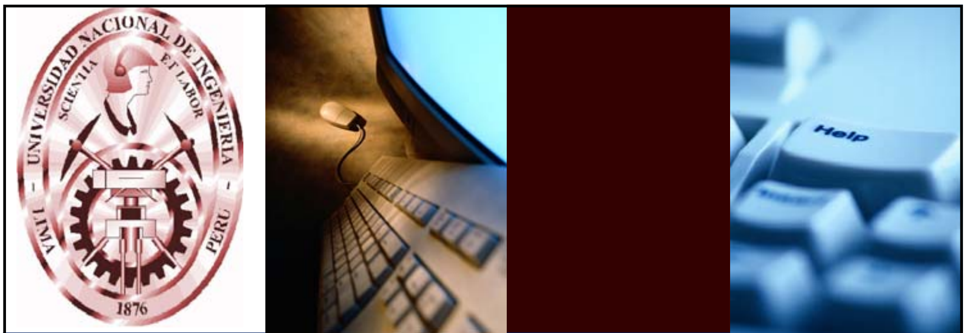
En T-2: $s_{T-1} = (1+r)(s_{T-2} - c_{T-2})$

$c_{T-1} = \beta(1+r)c_{T-2}$

$(1+\beta)c_T = (1+r)(s_{T-2} - c_{T-2})$

$$c_{T-2} = \frac{s_{T-2}}{1+\beta+\beta^2}$$

En T-i: $$c_{T-i} = \frac{s_{T-i}}{1+\beta+\beta^2+...+\beta^i}$$



Teoría Macrodinámica

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